CONSTRUCTION OF A GENERAL SOLUTION TO A SYSTEM OF MULTIGROUP TRANSPORT EQUATIONS

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One form of proof is discussed for the theorem on the completeness of the eigenfunctions in the ranges (−1, 1) and (0, 1) of the angular variable. The results may be used to determine the critical size of a planar reactor and to solve Milne's problem in the multigroup approximation.

The completeness of the eigenfunctions of a system of multigroup transport equations has been discussed [1,2], but a very complicated method was used to regularize the system of singular integral equations used in proving the theorem. A simpler proof of the completeness is given below.

1. Eigenfunctions and eigenvalues. The treatment of [3] is followed here. One of the basic problems of neutron transport theory is to find the spatial and energy distributions, which are described approximately by a system of linearized Boltzmann equations on the assumption that the total macroscopic interaction cross section has a power-law dependence on the energy.

We write the system of multigroup transport equations as

$$\mu \frac{\partial \varphi_i(x,\mu)}{\partial x} + \sigma_i \varphi_i(x,\mu) = \sum_{j=1}^{N} c_{ij} \int_{-1}^{+1} \varphi_j(x,\mu')\,d\mu' \qquad (i=1,\ldots,N). \tag{1.1}$$

Here σ_i is the total macroscopic cross section for the interaction and μ is the projection for unit velocity vector on the x axis

$$C_{ij} = \frac{1}{2}\left(\sigma^{j\to i} + \nu_f \sigma_f^j \alpha^i\right),$$

where $\sigma^{j\to i}$, ν_f, σ_f^j, α^i are, respectively, the cross section for scattering (elastic and inelastic), the number of secondary neutrons, the fission cross section for the group, and the fission-neutron spectrum.

We make the change of variable $t = \mu/\sigma_i$ and set $\psi_i(x,t) = \sigma_i \varphi_i(x, t/\sigma_i)$ to get

$$t\frac{\partial \psi_i(x,t)}{\partial x} + \psi_i(x,t) = \sum_{j=1}^{N} c_{ij} \int_{-\vartheta_j}^{\vartheta_j} \psi_j(x,t')\,dt' \qquad \left(\vartheta_j = \frac{1}{\sigma_j}\right). \tag{1.2}$$

The solution is sought in the form

$$\psi_i(x,t) = \exp(-x/\nu)\,\Phi_i(\nu,t). \tag{1.3}$$

Substitution of (1.3) into (1.2) gives

$$(\nu - t)\,\Phi_i(\nu,t) = \nu H_i(\nu),$$
$$H_i(\nu) = \sum_{j=1}^{N} c_{ij} \int_{-\vartheta_j}^{\vartheta_j} \Phi_j(\nu,t')\,dt'. \tag{1.4}$$

From (1.4) we have

$$\Phi_i(\nu,t) = \frac{\nu H_i(\nu)}{\nu - t} + \lambda_i(\nu)\,\delta(\nu - t). \tag{1.5}$$

The definition of $H_i(\nu)$ yields

$$\sum_{j=1}^{N} [\delta_{ij} - \nu c_{ij} f_j(\nu)]\,H_j(\nu) = \sum_{j=1}^{N} c_{ij}\lambda_j(\nu)\,\chi_j(\nu),$$
$$f_j(\nu) = \int_{-\vartheta_j}^{\vartheta_j} \frac{dt}{\nu - t}, \qquad \chi_j(\nu) = \int_{-\vartheta_j}^{\vartheta_j} \delta(\nu - t)\,dt, \tag{1.6}$$
$$\chi_j(\nu) = 1, \quad \text{if } \nu \in (-\vartheta_j, \vartheta_j),$$
$$\chi_j(\nu) = 0, \quad \text{if } \nu \notin (-\vartheta_j, \vartheta_j).$$

The right side of (1.6) becomes zero if $\nu \notin -\vartheta_0, \vartheta_0$, where $\vartheta_0 = 1/\sigma_0$, $\sigma_0 = \min(\sigma_1, \ldots, \sigma_N)$.

From the condition of solubility,

$$\sum_{j=1}^{N} [\delta_{ij} - \nu c_{ij} f_j(\nu)]\,H_j(\nu) = 0, \tag{1.7}$$

we get the characteristic equation

$$\Omega(\nu) \equiv \det[\delta_{ij} - \nu c_{ij} f_j(\nu)] = 0 \tag{1.8}$$

for the eigenvalues ν_s. The corresponding eigenfunctions are

$$\Phi_i(\nu_s, t) = \frac{\nu_s H_i(\nu_s)}{\nu_s - t}. \tag{1.9}$$

The total number of roots of the characteristic equation may be found via the principle of the argument,

$$2M = \frac{1}{2\pi i}\oint_C d\ln\Omega(\nu) = \frac{1}{2\pi i}\left[\ln\frac{\Omega^+(\vartheta_0)}{\Omega^-(\vartheta_0)} - \ln\frac{\Omega^+(-\vartheta_0)}{\Omega^-(-\vartheta_0)}\right]. \tag{1.10}$$

Here $\Omega^{\pm}(\nu)$ are the limiting values of the piecewise-holomorphic function $\Omega(z)$; contour C encloses the section $(-\vartheta_0, \vartheta_0)$ along the real axis. The continuum of eigenvalues lies in that range; the corresponding eigenfunctions are given by (1.5). We shall show that the eigenfunctions of (1.5) and (1.9) form a complete system.

2. Theorem 2.1. A system of arbitrary functions $F_i(t)$ that satisfy Hölder's condition for $|t| \le 1/\sigma_0 = \vartheta_0$ may be represented uniquely in the form

$$F_i(t) = \sum_{s=1}^{2M} A_s \Phi_i(\nu_s, t) + \int_{-\vartheta_0}^{\vartheta_0} \frac{\nu H_i(\nu)\,d\nu}{\nu - t} + \lambda_i(t) \qquad (t \in (-\vartheta_0, \vartheta_0)). \tag{2.1}$$

In other words, the system of singular integral equations of (2.1) allows us to determine uniquely the 2M constants A_s and the N functions $H_i(\nu)$.

Proof. We eliminate $\lambda_i(t)$ via (1.7) to get

$$\sum_{j=1}^{N} [\delta_{ij} - \nu c_{ij} f_j(\nu)]\,H_j(\nu) + \sum_{j=1}^{N} c_{ij}\chi_j(\nu) \int_{-\vartheta_0}^{\vartheta_0} \frac{\nu' H_j(\nu')\,d\nu'}{\nu' - \nu} = \nu \sum_{j=1}^{N} c_{ij}\chi_j(\nu) F_j'(\nu),$$
$$\left(F_j'(\nu) = F_j(\nu) - \sum_{s=1}^{2M} A_s \Phi_i(\nu_s, \nu)\right). \tag{2.2}$$

Consider the function

$$N_i(z)=\frac{1}{2\pi i}\int_{-\vartheta_0}^{\vartheta_0}\frac{\nu H_j(\nu)\,d\nu}{\nu-z}\qquad\left(\vartheta_0=\frac{1}{\sigma_0}\right).$$

This function is analytic in the plane with its section along the real axis from $-\vartheta_0$ to ϑ_0 and vanishes at infinity; the limiting values are given by the formula

$$N_j^{\pm}(\nu)=\frac{1}{2\pi i}\int_{-\vartheta_0}^{\vartheta_0}\frac{\nu' H_j(\nu')\,d\nu'}{\nu'-\nu}\pm\frac{1}{2}\nu H_j(\nu).$$

This implies that

$$N_j^+(\nu)-N_j^-(\nu)=\nu H_j(\nu),$$

$$N_j^+(\nu)+N_j^-(\nu)=\frac{1}{\pi i}\int_{-\vartheta_0}^{\vartheta_0}\frac{\nu' H_j(\nu')\,d\nu'}{\nu'-\nu}. \tag{2.3}$$

Substitution of (2.2) into (2.3) gives

$$\sum_{j=1}^{N}[\Omega_{ij}^+(\nu)N_j^+(\nu)-\Omega_{ij}^-(\nu)N_j^-(\nu)]=\nu\sum_{j=1}^{N}c_{ij}\chi_j(\nu)F_j'(\nu). \tag{2.4}$$

Then the above formula gives

$$\sum_{j=1}^{N}\Omega_{ij}(z)N_j(z)=\frac{1}{2\pi}\sum_{j=1}^{N}c_{ij}\int_{-\vartheta_0}^{\vartheta_0}\frac{\nu\chi_j(\nu)F_j'(\nu)\,d\nu}{\nu-z}+P_i^{(k)}(z). \tag{2.5}$$

Here $P_i^{(k)}(z)$ is an arbitrary polynomial. Since $\lim\Omega_{ij}(z)=\text{const}$ for $z\to\infty$, the functions $N_j(z)$ that vanish at infinity are the solution to (2.5) for $P_i^{(k)}(z)=0$. The solution of the system exists if

$$\sum_{i,j=1}^{N}c_{ij}H_i^{\circ}(\nu_l)\int_{-\vartheta_0}^{\vartheta_0}\frac{\nu\chi_j(\nu)F_j'(\nu)\,d\nu}{\nu-\nu_l}=0$$

$$(l=1,2,3,\ldots,2M). \tag{2.6}$$

Here $H_i^{\circ}(\nu_l)$ is the solution of the homogeneous system conjugate to (2.5). From (2.6) we derive the 2M constants A_s; the $\lambda_i(t)$ may be found via (1.7). The theorem is now proved.

It has been shown [3,4] that the determination of critical size and albedo for a plane layer may be reduced to a boundary problem of the type of (2.4).

The situation is different as regards the solution of Milne's problem. Here the $F_i(t)$ are given in the range $(0,\vartheta_0)$, and the above formula cannot be used to solve the problem of (2.4) because the functions $N_j(z)$ and matrix elements $\Omega_{ij}(z)$ are analytic in different regions. Hence we have to consider the problem

$$N_k^+(\nu)-\sum_{j=1}^{N}G_{kj}N_j^-(\nu)=g_k(\nu),$$

$$G_{kj}(\nu)=\sum_{i=1}^{N}[\Omega_{ki}^{-1}(\nu)]^+\Omega_{ij}^-(\nu),$$

$$g_k(\nu)=\sum_{i,j=1}^{N}[\Omega_{ki}^{-1}(\nu)]^+c_{ij}\chi_j(\nu)F_j'(\nu).$$

However, no effective means of solving this is known.

Matrices (c_{ij}) and (Ω_{ij}) are triangular in relation to the moderation of neutrons by nuclei of low or medium weight, whereupon the above problem may be solved fairly simply.

3. Theorem 3.1. A system of arbitrary functions $F_i(t)$ that satisfy Holder's condition for $0\le t\le 1/\sigma_0=\vartheta_0$ may be represented uniquely as

$$F_i(t)=\sum_{s=1}^{M}A_s\Phi_i(\nu_s,t)+\int_0^{\vartheta_0}\frac{\nu H_i(\nu)}{\nu-t}d\nu+$$
$$+\lambda_i(t)\qquad(i=1,\ldots,N). \tag{3.1}$$

Proof. The above arguments are repeated to reduce (3.1) to

$$N_i^+(\nu)G_{ii}(\nu)-N_i^-(\nu)=$$
$$=\frac{\nu}{\Omega_{ii}^-(\nu)}\sum_{j=1}^{i}c_{ij}\chi_j(\nu)F_j'(\nu)\quad(i=1,\ldots,N), \tag{3.2}$$

$$G_{ii}(\nu)=\frac{\Omega_{ii}^+(\nu)}{\Omega_{ii}^-(\nu)}F_j',\quad(\nu)=F_j(\nu)-\sum_{s=1}^{M}A_s\Phi_j(\nu_s,\nu). \tag{3.3}$$

Here $N_j^{\pm}(\nu)$ are the limiting values of the piecewise holomorphic function

$$N_j(z)=\frac{1}{2\pi i}\int_0^{\vartheta_0}\frac{\nu H_j(\nu)\,d\nu}{\nu-z},\qquad\lim_{z\to\infty}N_j(z)=0. \tag{3.4}$$

Now (c_{ij}) is a triangular matrix, so the number of roots to (1.8) is

$$2M=\frac{1}{2}\sum_{j=1}^{N}\delta_{ij}\left[\ln\frac{\Omega_{ij}^+(\vartheta_0)}{\Omega_{ij}^-(\vartheta_0)}-\ln\frac{\Omega_{ij}^+(-\vartheta_0)}{\Omega_{ij}^-(-\vartheta_0)}\right].$$

Then from $\Omega_{ij}^{\pm}(-\nu)=\Omega_{ij}^{\pm}(\nu)$ we have

$$2M=\frac{1}{\pi i}\sum_{j=1}^{N}\delta_{ij}\ln\frac{\Omega_{ij}^+(\vartheta_0)}{\Omega_{ij}^-(\vartheta_0)}\qquad\left(\vartheta_0=\frac{1}{\sigma_0}\right). \tag{3.5}$$

The index in the conjugation problem equals the sum of the partial indices [5]

$$\varkappa=\sum_{m=1}^{N}\varkappa_m=\frac{1}{2\pi i}\sum_{i,m=1}^{N}\delta_{mi}\left[\ln\frac{\Omega_{im}^+(0)}{\Omega_{im}^-(0)}-\ln\frac{\Omega_{im}^+(\vartheta_0)}{\Omega_{im}^-(\vartheta_0)}\right]=$$
$$=\frac{1}{2\pi i}\sum_{i,m=1}^{N}\delta_{mi}\ln\frac{\Omega_{im}^+(\vartheta_0)}{\Omega_{im}^-(\vartheta_0)}=-M. \tag{3.6}$$

Comparison of (3.5) and (3.6) shows that all the partial indices are negative, so the solution to (3.2) that vanishes at infinity is given by

$$N_m(z)=\frac{1}{2\pi iX_{mm}(z)}\int_0^{\vartheta_0}\frac{X_{mm}^-(\nu)}{\Omega_{mm}^-(\nu)}\left\{\nu\sum_{j=1}^{m}c_{mj}\chi_j(\nu)F_j'(\nu)-\right.$$
$$\left.-\sum_{j=1}^{m-1}[\Omega_{mj}^+(\nu)N_j^+(\nu)-\Omega_{mj}^-(\nu)N_j^-(\nu)]\right\}\frac{d\nu}{\nu-z}, \tag{3.7}$$

subject to the condition

$$\int_0^{\vartheta_0}\nu^{k_m}\frac{X_{mm}^-(\nu)}{\Omega_{mm}^-(\nu)}\left\{\nu\sum_{j=1}^{m}c_{mj}\chi_j(\nu)F_j'(\nu)-\right.$$
$$\left.-\sum_{j=1}^{m-1}[\Omega_{mj}^+(\nu)N_j^+(\nu)-\Omega_{mj}^-(\nu)N_j^-(\nu)]\right\}d\nu=0,$$
$$(k_m=0,1,\ldots,\varkappa_m-1). \tag{3.8}$$

Here $X_{mm}(z)$ is the solution to the homogeneous problem

$$X_{mm}^+(\nu)=G_{mm}(\nu)X_{mm}^-(\nu).$$

Hence the total number of additional conditions that define the A_s is equal to the total index.

We set m = 1 to find $N_1(z)$, which is substituted into the second equation of (3.7) to $N_2(z)$, and so on.

$H_i(\nu)$ is readily found via (2.3); (1.7) relates $\lambda_i(\nu)$ to $H_i(\nu)$.

The results are readily generalized to the case of anisotropic scattering.

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